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Note

An Improved Polynomial Representation for the Delta Function, $\delta(\mu - \mu_0)$

In numerical studies, it is often convenient to represent the delta function on the interval [−1, 1] as a series of Legendre polynomials. The usual representation is [1]

$$\delta_N(\mu, \mu_0) = \sum_{l=0}^{N} \left(\frac{2l+1}{2}\right) P_l(\mu)\, P_l(\mu_0), \tag{1}$$

which converges to a delta function on the interval [−1, 1] as N approaches infinity. However, there are several disadvantages to this representation. First, and most important, the convergence of $\delta_N(\mu, \mu_0)$ to $\delta(\mu - \mu_0)$ is very nonuniform in μ_0. Second, the functions δ_N are not positive definite, which can be a problem in certain situations.

We first discuss the rates of convergence of the family δ_N and then present what we feel to be an improved representation for $\delta(\mu - \mu_0)$ which is positive definite and yields more nearly uniform convergence.

The convergence of δ_N. To show how nonuniform the convergence of δ_N can be, we compute the ratios $\delta_N(1, 1)/\delta_N(1, 0)$ and $\delta_N(0, 0)/\delta_N(1, 0)$, both of which must converge to infinity as N approaches infinity. We employ Stirling's formula and find

$$P_l(1) = 1,$$

$$P_{2l}(0) = (-1)^l \frac{(2l)!}{2^{2l}(l!)^2} \approx (-1)^l \left(\frac{1}{\pi l}\right)^{1/2}, \qquad l \gg 1. \tag{2}$$

We now use Eq. (2) in the definition of δ_N to obtain the results

$$|\,\delta_N(1, 1)| = \left| \sum_{l=0}^{N} \frac{2l+1}{2} \right| \sim \frac{N^2}{2}, \tag{3a}$$

$$|\,\delta_N(0, 0)| \cong \left| \sum_{l=0}^{\alpha-1} \frac{4l+1}{2} P_{2l}^2(0) \right| + \left| \sum_{l=\alpha}^{N/2[1]} \frac{4l+1}{2} \frac{1}{\pi l} \right|$$

$$\sim A + \left| \frac{2}{\pi} \sum_{l=\alpha}^{N/2} 1 \right| + \left| \frac{1}{2\pi} \sum_{l=\alpha}^{N/2} \frac{1}{l} \right| \sim \frac{N}{\pi} \tag{3b}$$

[1] In Eqs. (3), fractional integers represent the appropriate integer parts of the fractions; i.e., $\alpha/2$ is the greatest integer less than $\alpha/2$.

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for $N \gg 2\alpha$, $\alpha \gg 1$, where A is independent of N because α is determined by the condition that the error in Stirling's formula be small. Similarly we obtain

$$\begin{aligned} |\,\delta_N(0, 1)| &= |\,\delta_N(1, 0)| \\ &\cong \left| \sum_{l=0}^{\alpha-1} \frac{4l+1}{2} P_{2l}(0)\, P_{2l}(1) \right| + \left| \sum_{l=\alpha}^{N/2} \frac{4l+1}{2} (-1)^l \left(\frac{1}{\pi l}\right)^{1/2} \right| \\ &= A' + \frac{1}{\pi^{1/2}} \left| \sum_{l=\alpha}^{N/2} (-1)^l \left(2l^{1/2} + \frac{1}{2l^{1/2}}\right) \right| . \end{aligned}$$

Now we group the terms at $l = 2m$ and $l = 2m + 1$ together with the result

$$\begin{aligned} |\,\delta_N(0, 1)| &\cong A' + \frac{1}{\pi^{1/2}} \left| \sum_{m=\alpha/2}^{N/4} \left(2(2m)^{1/2} + \frac{1}{2(2m)^{1/2}} \right.\right. \\ &\qquad \left.\left. - 2(2m+1)^{1/2} - \frac{1}{2(2m+1)^{1/2}}\right) \right| \\ &= A' + \frac{1}{\pi^{1/2}} \left| \sum_{m=\alpha/2}^{N/4} \left(2(2m)^{1/2} \left(1 - \left(1 + \frac{1}{2m}\right)^{1/2}\right) \right.\right. \\ &\qquad \left.\left. + \frac{1}{2(2m)^{1/2}} \left(1 - \left(1 + \frac{1}{2m}\right)^{-1/2}\right)\right) \right| \end{aligned}$$

or, using $m \gg 1$,

$$\cong A' + \frac{1}{\pi^{1/2}} \left| \sum_{m=\alpha/2}^{N/4} \frac{-1}{(2m)^{1/2}} + \frac{1}{2(2m)^{3/2}} \right| \sim \frac{1}{\pi^{1/2}} \left(\frac{N}{2}\right)^{1/2}, \tag{3c}$$

again for $N \gg 2\alpha$, $\alpha \gg 1$ with A' independent of N. (In Eq. (3c), we have changed the sums into integrals to evaluate the estimates.) Therefore, our ratios are

$$\begin{aligned} |\,\delta_N(1, 1)/\delta_N(0, 1)| &\sim N^{3/2}, \\ |\,\delta_N(0, 0)/\delta_N(1, 0)| &\sim N^{1/2}. \end{aligned} \tag{4}$$

Thus we see that δ_N converges much more rapidly for $\mu_0 = 1$ than for $\mu_0 = 0$.

An improved representation. We start by listing several properties we would like an improved representation $\tilde{\delta}_N$ to have:

(i) $\tilde{\delta}_N(\mu, \mu_0) \to \delta(\mu - \mu_0)$ more nearly uniformly in μ_0,

(ii) $\tilde{\delta}_N(\mu, \mu_0) \geqslant 0$,

(iii) The angular half-width of $\tilde{\delta}_N$ should be approximately independent of μ_0.

(iv) $\tilde{\delta}_N$ must be expressible in terms of P_0, P_N.

Condition (iii) requires some explanation. In many physical problems the angular half-width of the source is the relevant parameter, not its width in μ; therefore, we adopt condition (iii).

To satisfy these conditions we attempt a representation of $\delta(\mu - \mu_0)$ in terms of Chebyshev polynomials [1], $T_n(\mu) = \cos n(\cos^{-1} \mu)$. A brief calculation with geometric series results in

$$\begin{aligned} F_N &= T_0(\mu) + 2 \sum_1^{N-1} T_n(\mu_0)\, T_n(\mu) + T_N(\mu_0)\, T_N(\mu) \\ &= \frac{1}{2} \sin N(\theta + \theta_0) \cot \frac{(\theta + \theta_0)}{2} + \frac{1}{2} \sin N(\theta - \theta_0) \cot \frac{(\theta - \theta_0)}{2}, \end{aligned} \tag{5}$$

$\theta = \cos^{-1}(\mu)$, $\theta_0 = \cos^{-1}(\mu_0)$, which does converge to $\delta(\mu - \mu_0)$ (to within a constant factor). Furthermore, it is easy to show that F_N converges approximately uniformly in μ_0 and has approximately constant angular width. However, it is not positive definite and it only converges as N to $\delta(\mu - \mu_0)$.

We may eliminate both of these problems by averaging F_N over N and then subtracting off part of the result. We define

$$\begin{aligned} C_N \tilde{\delta}_N &= \sum_0^{N-1} F_n + \tfrac{1}{2} T_N(\mu)\, T_N(\mu_0) - \tfrac{1}{2} T_0(\mu)\, T_0(\mu_0) \\ &= \sin^2 \frac{N(\theta + \theta_0)}{2} \cot^2 \frac{(\theta + \theta_0)}{2} + \sin^2 \frac{N(\theta - \theta_0)}{2} \cot^2 \frac{(\theta - \theta_0)}{2} \\ &= (N - \tfrac{1}{2})\, T_0(\mu)\, T_0(\mu_0) + \sum_{n=1}^{N-1} 2(N - n)\, T_n(\mu)\, T_n(\mu_0) + \tfrac{1}{2} T_N(\mu)\, T_N(\mu_0). \end{aligned} \tag{6}$$

Examining the center line of (6) it is easy to see that $\tilde{\delta}_N$ converges to $\delta(\mu - \mu_0)$ as N^2. In addition, it is clear that $\tilde{\delta}_N$ is positive definite.

If faster convergence is required, subsequent averages may be performed on $\tilde{\delta}_N$, i.e.,

$$C_N^m \tilde{\delta}_N^m = \sum_{n=0}^{N} \tilde{\delta}_n^{m-1}. \tag{7}$$

It can easily be seen that $\tilde{\delta}_N^m$ converges to $\delta(\mu - \mu_0)$ as $N^{m+2}/(m + 2)!$. In the Appendix we give a simple, fast, method for representing the T_n in terms of Legendre polynomials.

Summary. We have constructed an improved representation for $\delta(\mu - \mu_0)$ which is positive definite, has approximately uniform convergence, and has approximately constant angular width.

Appendix

Here we show how to represent T_n in terms of P_l. Consider

$$T_n(\mu) = \sum_{l=0}^{n} C_{n,l} P_l(\mu). \tag{A1}$$

First note that odd n only couples with odd l and vice versa; therefore instead of needing a 40×40 array to store the $C_{n,l'}$ only two arrays 20×20 are needed, halving the storage space required. Using the differential equations for T_n, P_l and the standard recursion relations, one obtains

$$(n^2 - l^2)\, C_{n,l} + (2l + 1) \sum_{r=0}^{[(n-l-2)/2]} C_{n,l+2+2r} = 0, \tag{A2}$$

which determines all of the $C_{n,l}$ except $C_{n,n}$. These may be determined by matching the coefficients of the highest power of μ in P_n and T_n, yielding

$$C_{0,0} = 1, \qquad C_{1,1} = 1, \qquad \text{and} \qquad C_{n,n} = 2^{n-1} 2^n \frac{(n!)^2}{(2n)!}. \tag{A3}$$

Thus, all of the $C_{n,l}$ may be determined without calculating the coefficients of all the P_l, T_n or performing a large number of integrals.

Reference

1. W. W. Bell, "Special Functions for Scientists and Engineers," pp. 42–58, 187, 193, Van Nostrand, London, 1968.

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